$\boxed{S\ II}$

1. $A(x) = \begin{pmatrix} 0 & 0 & -1 \\ x & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

a) $A(1) + A(-1) = 2A(0) \overset{?}{\Rightarrow} \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}$ (A)

b) $A(x) + I_3 = \begin{pmatrix} 0 & 0 & -1 \\ x & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ x & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

$\det(A(x) + I_3) = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1 + 0 + x - 0 - 0 - 0 = 1 + x$

$1 + x = 0 \Rightarrow x = -1.$

c) $\det(aI_3 - bA(-1) + cA(-1)\cdot A(-1)) \geq 0$

$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} - \begin{pmatrix} 0 & 0 & -b \\ -b & 0 & 0 \\ 0 & -b & 0 \end{pmatrix} + c \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{pmatrix} + \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & c \\ c & 0 & 0 \end{pmatrix}}_{X}$ (1)

$A(-1)\cdot A(-1) = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

(1) $\det X = \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & a & c \\ c & b & a \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & a & c \\ c & b & a \end{vmatrix} =$

$= (a+b+c)(a^2 + c^2 + b^2 - ac - bc - ab) =$

$= \frac{1}{2}(a+b+c)\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \geq 0 \quad a, b, c \in \mathbb{R}_+.$